

1. [Simultaneous Equations Concepts -- Distance, Rate and Time](#)
2. [Simultaneous Equations Concepts -- Simultaneous Equations by Graphing](#)
3. [Simultaneous Equations Concepts -- Substitution](#)
4. [Simultaneous Equations Concepts -- Elimination](#)
5. [Simultaneous Equations Concepts -- Special Cases](#)
6. [Simultaneous Equations Concepts -- Word Problems](#)
7. [Simultaneous Equations Concepts -- Using Letters as Numbers](#)

Simultaneous Equations Concepts -- Distance, Rate and Time

If you travel 30 miles per hour for 4 hours, how far do you go? A little common sense will tell you that the answer is 120 miles.

This relationship is captured in the following equation:

where...

- is distance traveled (sometimes the letter x is used instead, for position)
- is the rate, or speed (sometimes the letter v is used, for velocity)
- is the time

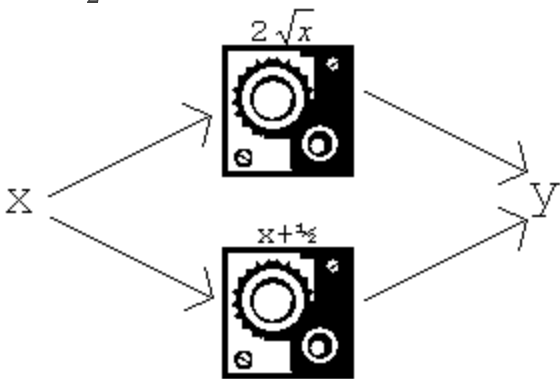
This is presented here because it forms the basis for many common simultaneous equations problems.

Simultaneous Equations Concepts -- Simultaneous Equations by Graphing

Consider the equation $y = 2\sqrt{x}$. How many (x,y) pairs are there that satisfy this equation? Answer: $(0,0)$, $(1,2)$, $(4,4)$, and $(9,6)$ are all solutions; and there is an **infinite number of other solutions**. (And don't forget non-integer solutions, such as $(,1)$!)

Now, consider the equation $y = x + \frac{1}{2}$. How many pairs satisfy **this** equation? Once again, an infinite number. Most equations that relate two variables have an infinite number of solutions.

To consider these two equations “simultaneously” is to ask the question: what (x,y) pairs make **both equations true**? To express the same question in terms of functions: what values can you hand the functions $2\sqrt{x}$ and $x + \frac{1}{2}$ that will make these two functions produce the **same answer**?

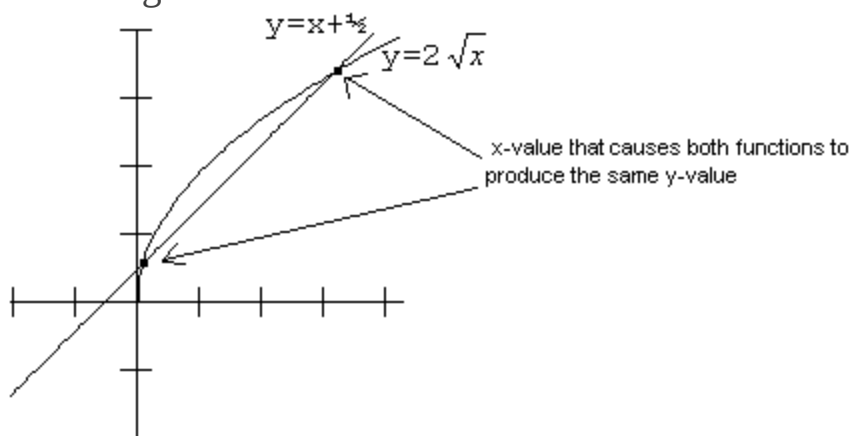


What number goes into **both functions** and makes them give the **same answer**? Is there even such a number? Is there more than one such number?

At first glance, it is not obvious how to approach such a question-- it is not even obvious how many answers there will be.

One way to answer such a question is by graphing. Remember, the graph of $y = 2\sqrt{x}$ is the set of all points that satisfy that relationship; and the graph of $y = x + \frac{1}{2}$ is the set of all points that satisfy that relationship. So **the intersection(s) of these two graphs is the set of all points that satisfy both relationships.**

How can we graph these two? The second one is easy: it is a line, already in $y = mx + b$ format. The y -intercept is $\frac{1}{2}$ and the slope is 1. We can graph the first equation by plotting points; or, if you happen to know what the graph of $y = \sqrt{x}$ looks like, you can stretch the graph vertically to get $y = 2\sqrt{x}$, since all the y -values will double. Either way, you wind up with something like this:



We can see that there are two points of intersection. One occurs when x is barely greater than 0 (say, $x = 0.1$), and the other occurs at approximately $x = 3$. There will be no more points of intersection after this, because the line will rise faster than the curve.

Exercise:

Problem: $y = 2\sqrt{x}$

$$y = x + \frac{1}{2}$$

Solution:

From graphing...

$$x = 0.1, x = 3$$

Graphing has three distinct advantages as a method for solving simultaneous equations.

1. It works on any type of equations.
2. It tells you **how many** solutions there are, as well as what the solutions are.
3. It can help give you an **intuitive feel** for why the solutions came out the way they did.

However, graphing also has two **disadvantages**.

1. It is time-consuming.
2. It often yields solutions that are **approximate**, not exact—because you find the solutions by simply “eyeballing” the graph to see where the two curves meet.

For instance, if you plug the number 3 into both of these functions, will you get the same answer?

$$3 \rightarrow 2\sqrt{x} \rightarrow 2\sqrt{3} \approx 3.46$$

$$3 \rightarrow x + \frac{1}{2} \rightarrow 3.5$$

Pretty close! Similarly, $2\sqrt{.1} \approx 0.632$, which is quite close to 0.6. But if we want more exact answers, we will need to draw a much more exact graph, which becomes **very** time-consuming. (Rounded to three decimal places, the actual answers are 0.086 and 2.914.)

For more exact answers, we use analytic methods. Two such methods will be discussed in this chapter: **substitution** and **elimination**. A third method will be discussed in the section on Matrices.

Simultaneous Equations Concepts -- Substitution

Here is the algorithm for substitution.

1. Solve one of the equations for one variable.
2. Plug this variable into the other equation.
3. Solve the second equation, which now has only one variable.
4. Finally, use the equation you found in step (1) to find the other variable.

Example:

Solving Simultaneous Equations by Substitution

$$3x + 4y = 1$$

$$2x - y = 8$$

1. The easiest variable to solve for here is the y in the second equation.

- $-y = -2x + 8$
- $y = 2x - 8$

2. Now, we plug that into the **other** equation:

- $3x + 4(2x - 8) = 1$

3. We now have an equation with only x in it, so we can solve for x .

- $3x + 8x - 32 = 1$
- $11x = 33$
- $x = 3$

4. Finally, we take the equation from step (1), $y = 2x - 8$, and use it to find y .

- $y = 2(3) - 8 = -2$

So $(3, -2)$ is the solution. You can confirm this by plugging this pair into both of the original equations.

Why does substitution work?

We found in the first step that $y = 2x - 8$. This means that y and $2x - 8$ are **equal** in the sense that we discussed in the first chapter on functions—they will always be the same number, in these equations—they are the **same**. This gives us permission to simply replace one with the other, which is what we do in the second (“substitution”) step.

Simultaneous Equations Concepts -- Elimination

Here is the algorithm for elimination.

1. Multiply one equation (or in some cases both) by some number, so that the two equations have the **same coefficient** for one of the variables.
2. Add or subtract the two equations to make that variable go away.
3. Solve the resulting equation, which now has only one variable.
4. Finally, plug back in to find the other variable.

Example:

Solving Simultaneous Equations by Elimination

$$3x + 4y = 1$$

$$2x - y = 8$$

- **1**The first question is: how do we get one of these variables to have the **same coefficient in both equations**? To get the x coefficients to be the same, we would have to multiply the top equation by 2 and the bottom by 3. It is much easier with y ; if we simply multiply the bottom equation by 4, then the two y values will both be multiplied by 4.
 - $3x + 4y = 1$
 - $8x - 4y = 32$
- **2**Now we either **add** or **subtract** the two equations. In this case, we have $4y$ on top, and $-4y$ on the bottom; so if we add them, they will cancel out. (If the bottom had $+4y$ we would have to subtract the two equations to get the " y "s to cancel.)
 - $11x + 0y = 33$
- **3-4** Once again, we are left with only one variable. We can solve this equation to find that $x = 3$ and then plug back in to either of the original equations to find $y = -2$ as before.

Why does elimination work?

As you know, you are always allowed to do the **same thing** to both sides of an equation. If an equation is true, it will still be true if you add 4 to both sides, multiply both sides by 6, or take the square root of both sides.

Now—consider, in the second step above, what we did to the equation $3x + 4y = 1$. We **added** something to both sides of this equation. What did we add? On the left, we added $8x - 4y$; on the right, we added 32. It seems that we have done something **different** to the two sides.

However, the second equation gives us a guarantee that these two quantities, $8x - 4y$ and 32, are in fact **the same as each other**. So by adding $8x - 4y$ to the left, and 32 to the right, we really have done exactly the same thing to both sides of the equation $3x + 4y = 1$.

Simultaneous Equations Concepts -- Special Cases

This module discusses concepts related to simultaneous equations.

Consider the two equations:

Equation:

$$2x + 3y = 8$$

Equation:

$$4x + 6y = 3$$

Suppose we attempt to solve these two equations by elimination. So, we double the first equation and subtract, and the result is:

Equation:

$$4x + 6y = 16$$

$$4x + 6y = 3$$

$$0 = 13$$

Hey, what happened? 0 does not equal 13, no matter what x is.

Mathematically, we see that these two equations have **no** simultaneous solution. You asked the question “When will both of these equations be true?” And the math answered, “Hey, buddy, not until 0 equals 13.”

No solution.

Now, consider these equations:

Equation:

$$2x + 3y = 8$$

$$4x + 6y = 16$$

Once again, we attempt elimination, but the result is different:

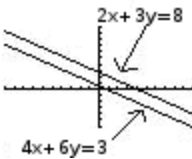
Equation:

$$\begin{aligned}
 2x + 3y &= 8 \\
 4x + 6y &= 16 \\
 0 &= 0
 \end{aligned}$$

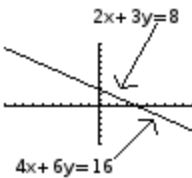
What happened **that** time? $0 = 0$ no matter what x is. Instead of an equation that is always false, we have an equation that is always true. Does that mean these equations work for **any** x and y ? Clearly not: for instance, $(1,1)$ does not make either equation true. What this means is that the two equations are the same: any pair that solves one will also solve the other. There is an infinite number of solutions.

Infinite number of solutions.

All of this is much easier to understand graphically! Remember that one way to solve simultaneous equations is by graphing them and looking for the intersection. In the first case, we see that original equations represented two **parallel lines**. There is no point of intersection, so there is no simultaneous equation.



In the second case, we see that the original equations represented the **same line, in two different forms**. Any point on the line is a solution to both equations.



Note: If you solve an equation and get a mathematical impossibility such as $0 = 13$, there is no solution. If you get a mathematical tautology such as $0 = 0$, there is an infinite number of solutions.

Simultaneous Equations Concepts -- Word Problems

Many students approach math with the attitude that “I can do the equations, but I’m just not a ‘word problems’ person.” No offense, but that’s like saying “I’m pretty good at handling a tennis racket, as long as there’s no ball involved.” The only point of handling the tennis racket is to hit the ball. The only point of math equations is to solve problems. So if you find yourself in that category, try this sentence instead: “I’ve never been good at word problems. There must be something about them I don’t understand, so I’ll try to learn it.”

Actually, many of the key problems with word problems were discussed in the very beginning of the “Functions” unit, in the discussion of variable descriptions. So this might be a good time to quickly re-read that section. If you can correctly identify the variables, you’re half-way through the hard part of a word problem. The other half is **translating the sentences of the problem into equations** that use those variables.

Let’s work through an example, very carefully.

Example:

Simultaneous Equation Word Problem

A roll of dimes and a roll of quarters lie on the table in front of you. There are three more quarters than dimes. But the quarters are worth three times the amount that the dimes are worth. How many of each do you have?

1. Identify and label the variables.

- There are actually two different, valid ways to approach this problem. You could make a variable that represents the number of dimes; or you could have a variable that represents the value of the dimes. Either way will lead you to the right answer. However, it is vital to know which one you’re doing! If you get confused half-way through the problem, you will end up with the wrong answer.

Let's try it this way:

d is the number of dimes

q is the number of quarters

2. Translate the sentences in the problem into equations.

- "There are three more quarters than dimes" $\rightarrow q = d + 3$
- "The quarters are worth three times the amount that the dimes are worth" $\rightarrow 25q = 3(10d)$
- This second equation relies on the fact that if you have q quarters, they are worth a total of $25q$ cents.

3. Solve.

- We can do this by elimination or substitution. Since the first equation is already solved for q , I will substitute that into the second equation and then solve.

$$25(d + 3) = 3(10d)$$

$$25d + 75 = 30d$$

$$75 = 5d$$

$$d = 15$$

$$q = 18$$

So, did it work? The surest check is to go all the way back to the original problem—not the equations, but the words. We have concluded that there are 15 **dimes** and **18** quarters.

“There are three more quarters than dimes.”

“The quarters are worth three times the amount that the dimes are worth.”

→ Well, the quarters are worth $18 \cdot 25 = \$4.50$. The dimes are worth $15 \cdot 10 = \$1.50$.

Simultaneous Equations Concepts -- Using Letters as Numbers

This module shows how it can be helpful, on occasion, to use letters as numbers in order to quickly find solutions to variations on a simultaneous equation problem.

Toward the end of this chapter, there are some problems in substitution and elimination where **letters are used in place of numbers**. For instance, consider the following problem:

Equation:

$$2y - ax = 7$$

Equation:

$$4y + 3ax = 9$$

What do we do with those "a"s? Like any other variable, they simply represent an unknown number. As we solve for x , we will simply leave a as a variable.

This problem lends itself more naturally to elimination than to substitution, so I will double the top equation and then subtract the two equations and solve.

Equation:

$$\begin{array}{r} 4y - 2ax = 14 \\ -(4y + 3ax = 9) \\ \hline 0y - 5ax = 5 \end{array}$$

Equation:

$$x = \frac{5}{-5a} = \frac{-1}{a}$$

As always, we can solve for the second variable by plugging into either of our original equations.

Equation:

$$2y - a\left(\frac{-1}{a}\right) = 7$$

Equation:

$$2y + 1 = 7$$

Equation:

$$y = 3$$

There is no new math here, just elimination. The real trick is not to be spooked by the a , and do the math just like you did before.

And what does that **mean**? It means we have found a solution that works for those two equations, regardless of a . We can now solve the following three problems (and an infinite number of others) without going through the hard work.

If $a = 5$,	If $a = 10$,	If $a = -3$,
The original equations become:	The original equations become:	The original equations become:
$2y - 5x = 7$ $4y + 15x = 9$	$2y - 10x = 7$ $4y + 30x = 9$	$2y + 3x = 7$ $4y - 9x = 9$
And the solution is:	And the solution is:	And the solution is:

If $a = 5$,	If $a = 10$,	If $a = -3$,
$x = \frac{-1}{5} \quad y = 3$	$x = \frac{-1}{10} \quad y = 3$	$x = \frac{1}{3} \quad y = 3$

The whole point is that I did not have to **solve** those three problems—by elimination, substitution, or anything else. All I had to do was plug a into the general answer I had already found previously. If I had to solve a hundred such problems, I would have saved myself a great deal of time by going through the hard work once to find a general solution!

Mathematicians use this trick all the time. If they are faced with many similar problems, they will attempt to find a **general** problem that encompasses all the **specific** problems, by using variables to replace the numbers that change. You will do this in an even more general way in the text, when you solve the “general” simultaneous equations where **all** the numbers are variables. Then you will have a formula that you can plug any pair of simultaneous equations into to find the answer at once. This formula would also make it very easy, for instance, to program a computer to solve simultaneous equations (computers are terrible at figuring things out, but they’re great at formulas).